ON TRANSCENDENTAL NUMBERS*

BY

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In 1851, Liouville† gave the following theorem (the proof of which is very simple):

Let $f(z) = 0$ be an equation of degree $n \geqq 2$ with real integral coefficients and irreducible in the domain $R(1)$, z a real root of this equation, and p/q any real rational number. Then a positive number A can be found, which is independent of n, such that for all p/q,

$$\left| z - \frac{p}{q} \right| > \frac{1}{Aq^n}.$$

By applying his theorem to numbers of the form $\sum_{\nu=1}^{\infty} (\alpha_\nu / l^{\nu!})$, l a positive integer $\geqq 2$, α_ν any integer whose absolute value is limited, Liouville showed that all such numbers are transcendental.

By choosing $l = 10$, a rule is obtained for representing in decimal-fraction form the numbers contained in a certain non-enumerable set of transcendental numbers.‡

From Liouville's method of proof it follows that all numbers $\sum_{\nu=0}^{\infty} (\alpha_\nu / l^{\gamma_\nu})$, when only γ_ν increases "sufficiently rapidly," are transcendental.

E. Maillet§ proves in his book the transcendency of the members of certain sets of numbers, the simplest of which are represented by series of the type

$$\sum_{n=0}^{\infty} (a_n \div b^{nb^{b^n}}) \cdot x^n$$

for all rational and even for all algebraic x, and G. Faber|| uses a generalization of Liouville's theorem and treats a more general type of series which may

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† Journal de mathématiques pures et appliquées, vol. 16 (1851), p. 133.

‡ The first proof of the existence of transcendental numbers was also given by Liouville, in the Comptes-rendus, vol. 18 (1844), p. 883, p. 910 (reproduced in †), and was based on an investigation of continued fractions.

§ *Introduction à la théorie des nombres transcendants et des propriétés arithmétiques des fonctions*, Paris (1906), p. 105.

|| Mathematische Annalen, vol. 58 (1904), p. 545.

by this method be shown to yield transcendental numbers, namely all series of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{h_n \cdot x^n}{(p_1 \cdot p_2 \cdots p_n)^{q_1 \cdot q_2 \cdots q_n}},$$

where the h_n, p_n, q_n are integers, h_n finite (or growing infinite with n in some particular fashion),

$$p_n > 1, \qquad \lim_{n=\infty} q_n = \infty;$$

and he shows that $f(x)$ is a transcendental number for all algebraic x. These seem to be about the only advances along this line.* Series of the type $\sum_{n=0}^{\infty} (\alpha_n/a^{b^n})$ can not be treated by these methods.

The object of the present paper is to prove the following

THEOREM. *Let a be an integer greater than* 1, p/q *a rational fraction,* $p \gtrless 0$, $q > 0$; α_n, $(n = 0, 1, 2, \cdots)$, *any positive or negative integer smaller in absolute value than a fixed arbitrary number* M, *but only a finite number of the* α_n *equal to* 0; *then* $\sum_{n=0}^{\infty} (\alpha_n/a^{2^n}) \cdot (p/q)^n$ *is always a transcendental number.*

We shall prove the theorem in the form just stated. However, the proof still holds, with only slight modifications, for either or both of the following extensions:

1. The exponent 2^n may be replaced everywhere by b^n, b any fixed positive integer greater than 2.

2. α_n need not be limited; for example, the number is transcendental when $|\alpha_n| < R^n$, R an arbitrary fixed number; or α_n may be a rational number δ/ϵ, $|\delta| < R^n$, $\epsilon < R^n$, but always with the restriction that only a finite number of the α_n equal to 0.

Proof. All letters to be used will denote real numbers and all are integers except z and c_3. The symbol r^{s^t} means $r^{(s^t)}$, $r^{s^{t^u}}$, means $r^{(s^{t^u})}$, etc.

Max. $(a_1, a_2, a_3, \cdots, a_\tau)$ denotes the largest of the positive values $a_1, a_2, a_3, \cdots, a_\tau$.

Let $z = \sum_{n=0}^{\infty} (\alpha_n/a^{2^n}) \cdot (p/q)^n$, with the restrictions on the α_n, a, p, and q mentioned in the theorem.

If z is not a transcendental number, there must exist a certain equation

$$f(z) = \sum_{\mu=0}^{k} A_\mu z^\mu = 0,$$

where k is a fixed positive integer and the A_μ are integers, $A_k \neq 0$. Let N be a positive integer, such that $N > |A_\mu|$, $(\mu = 0, 1, \cdots, k)$. We shall

* E. Borel, *Leçons sur la théorie de la croissance, recueillies et redigées par* A. Denjoy, Paris (1910), Chapter V, and Axel Thue, Journal für Mathematik, vol. 135 (1909), p. 284, particularly the latter, have generalized Liouville's theorem in important respects, but these generalizations have so far not had any influence on the investigation of transcendental numbers.

substitute $z = \sum_{n=0}^{\infty}(\alpha_n/a^{2^n}) \cdot (p/q)^n$ in $f(z)$ and shall show that $f(z) \neq 0$ for any given $f(z)$.

Once chosen, all of the following numbers are to be considered constant: α_n $(n = 0, 1, \cdots)$; a; p; q; k; A_μ $(\mu = 0, 1, 2, \cdots, k)$; M; N.

Since $\sum_{n=0}^{\infty}(\alpha_n/a^{2^n}) \cdot x^n$ is a power-series, convergent for all values of x, we may substitute z in $f(z)$ and rearrange terms as we like. After substituting, we shall arrange the terms according to increasing denominators, *without canceling anywhere,* and collect terms with equal denominators.

The denominators are formed by taking the product of k or fewer factors of the form $a^{2^\nu} \cdot q^\nu$, repetition admitted.

We prove first the following statement:

When at the same time $n > k$ *and* $n > n_1$, *where* n_1 *is a certain positive integer which will be characterized in the proof, then the three numbers*

$$\gamma_1 = a^{2^{n-1}+2^{n-2}+\cdots+2^{n-k+1}+2^{n-k-1}} \cdot q^{(n-1)+(n-2)+\cdots+(n-k+1)+(n-k-1)},$$

$$\gamma_2 = a^{2^{n-1}+2^{n-2}+\cdots+2^{n-k+1}+2^{n-k}} \cdot q^{(n-1)+(n-2)+\cdots+(n-k+1)+(n-k)},$$

$$\gamma_3 = a^{2^n} \cdot q^n$$

satisfy the two conditions:

(1) $$\gamma_1 < \gamma_2 < \gamma_3,$$

(2) $$\gamma_1, \gamma_2, \gamma_3$$

are three consecutive denominators of our fractions in $\sum_{\mu=0}^{k} A_\mu z^\mu$.

It is clear that all denominators containing any factor a^{2^ν}, $\nu > n$, are larger than γ_3, and of all denominators containing the factor a^{2^n}, γ_3 is the smallest. Consequently all denominators smaller than γ_3 contain only factors $a^{2^\nu} \cdot q^\nu$, $(\nu = 0, 1, \cdots, n-1)$. Of all denominators smaller than γ_3 containing not more than k of these factors, $(n > k)$, it is obvious that there is none between γ_2 and γ_1 and none between γ_2 and γ_3. It remains however to be shown that $\gamma_2 < \gamma_3$, in spite of the higher powers of q involved in γ_2, that is:

$$a^{2^{n-1}+2^{n-2}+\cdots+2^{n-k}} \cdot q^{(n-1)+(n-2)+\cdots+(n-k)} < a^{2^n} \cdot q^n$$

for all n from a certain value on. We have

$$q^{(n-1)+(n-2)+\cdots+(n-k)-n} < a^{2^{n-k}},$$

which is *a fortiori* satisfied when

$$q^{n^2/2} = (q^{\frac{1}{2}})^{n^2} < a^{2^{n-k}} = (a^{2^{-k}})^{2^n};$$

this is true from a certain integer $n = n_1$ on. Thus our statement is proved, and we shall henceforth take n greater than Max (k, n_1).

To prove that $f(z) \neq 0$, we shall show that $f(z)$ may be written in the form

$$f(z) = \frac{c_1}{\gamma_1} + \frac{c_2}{\gamma_2} + c_3,$$

satisfying the following conditions:

(a) c_1, c_2 are integers, $c_1 \gtreqless 0, c_2 \neq 0$; c_3 a real number $\gtreqless 0$; γ_1, γ_2 numbers of the type defined above, so that γ_2 is a multiple of γ_1; $\gamma_2 = l \cdot \gamma_1, l > 1$.

(b) $\dfrac{|c_2|}{\gamma_2} < \dfrac{1}{\gamma_1}$.

(c) $|c_3| < \dfrac{1}{\gamma_2}$.

We must admit four possible cases:

If $c_1 = 0, c_3 = 0: f(z) \neq 0$ because $c_2 \neq 0$.

If $c_1 = 0, c_3 \neq 0: f(z) = \dfrac{c_2}{\gamma_2} + c_3 \neq 0$ from (a) and (c), without using (b).

If $c_1 \neq 0, c_3 = 0: f(z) = \dfrac{c_1}{\gamma_1} + \dfrac{c_2}{\gamma_2} \neq 0$ from (a) and (b), without using (c).

If $c_1 \neq 0, c_3 \neq 0: \dfrac{|c_1|}{\gamma_1} = \dfrac{l|c_1|}{\gamma_2} > \dfrac{c_2}{\gamma_2}$, therefore

$$\frac{|c_1|}{\gamma_1} - \frac{|c_2|}{\gamma_2} \geqq \frac{1}{\gamma_2} > |c_3|, \qquad \frac{|c_1|}{\gamma_1} > \frac{|c_2|}{\gamma_2} + |c_3|,$$

and hence

$$f(z) = \frac{c_1}{\gamma_1} + \frac{c_2}{\gamma_2} + c_3 \neq 0.$$

Always assuming that no cancellations have been performed, either in

$$\textstyle\sum_{n=0}^{\infty} (\alpha_n / a^{2^n}) \cdot (p/q)^n \quad \text{or in} \quad \sum_{\mu=0}^{k} A_\mu z^\mu,$$

our c_2/γ_2 shall consist of all terms which have exactly the denominator

$$\gamma_2 = a^{2^{n-1}+2^{n-2}+\cdots+2^{n-k}} \cdot q^{(n-1)+(n-2)+\cdots+(n-k)},$$

n being properly chosen. Evidently c_2/γ_2 arises entirely from the term $A_k z^k$ of $f(z)$, and we obtain

$$\frac{c_2}{\gamma_2} = \frac{A_k \cdot \alpha_{n-1} \cdot \alpha_{n-2} \cdots \alpha_{n-k} \cdot p^{(n-1)+(n-2)+\cdots+(n-k)}}{a^{2^{n-1}+2^{n-2}+\cdots+2^{n-k}} \cdot q^{(n-1)+(n-2)+\cdots+(n-k)}} \cdot \frac{k!}{1!\,1! \cdots 1!},$$

$$\frac{|c_2|}{\gamma_2} \leqq \frac{k!\,N \cdot M^k \cdot |p|^{(n-1)+(n-2)+\cdots+(n-k)}}{a^{2^{n-1}+2^{n-2}+\cdots+2^{n-k}} \cdot q^{(n-1)+(n-2)+\cdots+(n-k)}},$$

and $c_2 \neq 0$, provided $n > n_2$, where n_2 is so large that α_{n_2-k} and all following coefficients are different from zero.

Our c_1/γ_1 shall comprise all terms with denominators smaller than γ_2. If $c_1 = 0$, c_1/γ_1 is not needed. If $c_1 \neq 0$, then

$$\gamma_1 = a^{2^{n-1}+2^{n-2}+\cdots+2^{n-k+1}+2^{n-k-1}} \cdot q^{(n-1)+(n-2)+\cdots+(n-k+1)+(n-k-1)}$$

is common denominator of all fractions under consideration, which can therefore be added together, giving c_1/γ_1.

To prove $|c_2|/\gamma_2 < 1/\gamma_1$, we show that, from a certain $n = n_3$ on,

$$\frac{k!\,N \cdot M^k \cdot |p|^{(n-1)+(n-2)+\cdots+(n-k)}}{a^{2^{n-1}+2^{n-2}+\cdots+2^{n-k}} \cdot q^{(n-1)+(n-2)+\cdots+(n-k)}}$$

$$< \frac{1}{a^{2^{n-1}+2^{n-2}+\cdots+2^{n-k+1}+2^{n-k-1}} \cdot q^{(n-1)+(n-2)+\cdots+(n-k+1)+(n-k-1)}}.$$

We assume $n > \text{Max}\ (k, n_1, n_2)$. Our inequality reduces to

$$\frac{k!\,N \cdot M^k}{q} \cdot |p|^{(n-1)+(n-2)+\cdots+(n-k)} < a^{2^{n-k-1}},$$

which is *a fortiori* satisfied if

$$C_1 \cdot |p|^{n^2/2} < a^{2^{n-k-1}},$$

where

$$C_1 = \frac{k!\,N \cdot M^k}{q}$$

is independent of n;

$$C_1 \cdot |p^{\frac{1}{2}}|^{n^2} < (a^{2^{-k-1}})^{2^n},$$

which is true from a certain n_3 on. We assume $n > \text{Max}\ (k, n_1, n_2, n_3)$. This proves (*b*).

Our c_3 shall consist of all terms with denominators greater than γ_2, that is, of all terms of $f(z)$ not accounted for by $c_1/\gamma_1 + c_2/\gamma_2$. Hence c_3 is built up of fractions of the form

$$\frac{A_\mu \cdot \alpha_{\tau_1} \cdot \alpha_{\tau_2} \cdots \alpha_{\tau_\rho} \cdot p^{\tau_1+\tau_2+\cdots+\tau_\rho}}{a^{2^{\tau_1}+2^{\tau_2}+\cdots+2^{\tau_\rho}} \cdot q^{\tau_1+\tau_2+\cdots+\tau_\rho}},$$

where $\rho \leqq k$, and $\tau_1, \tau_2, \cdots, \tau_\rho$ are numbers of the sequence $0, 1, 2, \cdots$ not necessarily different from each other.

These denominators are all of the form $a^{\lambda_1} \cdot q^{\lambda_2}$, where the exponents of a are formed by taking the sum of k or less numbers of the infinite sequence 2^ν, $(\nu = 0, 1, 2, \cdots)$, repetition permitted, with the restriction that at least one of the k or fewer numbers 2^ν shall be greater than or equal to 2^n, where $n > \text{Max}\ (k, n_1, n_2, n_3)$. There are, as is easily seen, less than $(n+2)^k$ exponents that can be so formed from the first $n+1$ numbers $2^0, 2^1, \cdots, 2^n$, counting exponents separately even when they differ only in the order of their summands $2^{\tau_1}, 2^{\tau_2}, \cdots, 2^{\tau_\rho}$. In the same way it is seen that there are less than $(n+3)^k$ exponents formed by taking only numbers of the set $2^0, 2^1, \cdots, 2^{n+1}$; less than $(n+l+2)^k$ by taking only numbers of the set $2^0, 2^1, \cdots, 2^{n+l}$ for $l = 1, 2, 3, \cdots$. The denominators $a^{\lambda_1} \cdot q^{\lambda_2}$ all contain also factors q^{λ_2}. Hence there are certainly less than $(n+2)^k$ fractions with denominators smaller than a^{2^n}, less than $(n+3)^k$ fractions with denominators d, where $a^{2^n} \leqq d < a^{2^{n+1}}$, because there are altogether less than $(n+3)^k$ fractions with denominators less than or equal to $a^{2^{n+1}}$, and less than $(n+l+2)^k$ fractions with denominators d, where $a^{2^{n+l-1}} \leqq d < a^{2^{n+l}}$. I increase (or at least do not decrease) the absolute value of all fractions by replacing q^{λ_2} by 1.

Those numerators belonging to denominators d, $a^{2^n} \leqq d < a^{2^{n+1}}$, have all of their $\tau_1, \tau_2, \cdots, \tau_\rho$ not larger than n, those belonging to denominators d, $a^{2^{n+1}} \leqq d < a^{2^{n+2}}$, have all of their $\tau \leqq n+1$, and those fractions with denominators d, $a^{2^{n+l}} \leqq d < a^{2^{n+l+1}}$, have all $\tau \leqq n+l$.

Altogether we find for c_3:

$$|c_3| < N \cdot M^k \cdot \left[\frac{(n+2)^k \cdot |p|^{kn}}{a^{2^n}} + \frac{(n+3)^k \cdot |p|^{k(n+1)}}{a^{2^{n+1}}} + \cdots + \frac{(n+l+2)^k \cdot |p|^{k(n+l)}}{a^{2^{n+l}}} + \cdots \right].$$

The convergence of this expression is obvious. Besides taking n greater than Max. (k, n_1, n_2, n_3) we now take n so large that the sum in brackets is smaller than

$$\frac{2(n+2)^k \cdot |p|^{kn}}{a^{2^n}}.$$

This is certainly true when the ratio of two consecutive terms is always smaller than $\frac{1}{2}$, and happens, for example, when $n > \log_2 [k \cdot \log_a |4p|]$, as is easily verified. Let n_4 be an integer satisfying this relation, and take n greater than Max. (k, n_1, n_2, n_3, n_4). Substituting, we have

$$|c_3| < \frac{2N \cdot M^k \cdot (n+2)^k \cdot |p|^{kn}}{a^{2^n}},$$

and we shall show that the expression on the right is, for sufficiently large values of n, less than

$$\frac{1}{a^{2^{n-1}+2^{n-2}+\cdots+2^{n-k}} \cdot q^{(n-1)+(n-2)+\cdots+(n-k)}},$$

thus establishing the inequality $|c_3| < 1/\gamma_2$ and proving our theorem.

Our inequality for n reduces to

$$2N \cdot M^k \cdot (n+2)^k \cdot |p|^{kn} \cdot q^{(n-1)+(n-2)+\cdots+(n-k)} < a^{2^{n-k}},$$

which is *a fortiori* satisfied when

$$C_2 \cdot (n+2)^k \cdot |p^k|^n \cdot (q^{\frac{1}{2}})^{n^2} < (a^{2^{-k}})^{2^n},$$

where $C_2 = 2N \cdot M^k$ is independent of n.

By making n sufficiently large we can satisfy the following three inequalities simultaneously:

$$C_2 \cdot (n+2)^k < |p^k|^n, \qquad |p^{2k}|^n < (q^{\frac{1}{2}})^{n^2}, \qquad q^{n^2} < (a^{2^{-k}})^{2^n},$$

which, combined, prove our inequality for all n greater than a certain integer n_6. By taking n greater than Max. $(k, n_1, n_2, n_3, n_4, n_5, n_6)$ we meet all restrictions which have been successively imposed on n during the proof.

The condition that only a finite number of coefficients shall be zero (in order to ensure $c_2 \neq 0$), I have not been able to remove.

By taking $p/q = 1$, we see that all numbers $\sum_{n=0}^{\infty} (\alpha_n / a^{2^n})$, ($\alpha_n$ an integer, $|\alpha_n|$ limited, and from a certain point on $|\alpha_n| \geqq 1$) are transcendental.

As another special case we mention the function, $\sum_{n=0}^{\infty} x^{2^n}$, introduced by Fredholm* to demonstrate the existence of analytical functions possessing certain peculiar properties on their natural boundaries. It follows from our theorem that this function has transcendental values for an infinite set of real rational values of the argument having the origin as a limiting point.

* See Mittag-Leffler, Acta Mathematica, vol. 15 (1891), p. 279.

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